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Value distribution of finely harmonic morphisms and applications in complex analysis

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Introduction

The aim of this article is to generalize (e.g. to several complex variables) and in part to strengthen the following three classical results on removable singularities of meromorphic functions in which either the singularity set or the corresponding cluster set is *polar*, i.e. locally of zero outer (logarithmic) capacity.

Let U denote a domain in $\mathbb{C}$ (or on a Riemann surface), and F a relatively closed proper subset of U. A meromorphic function $\varphi : U\setminus F \to \mathbb{C}_\infty$ $(= \mathbb{C} \cup \{\infty\})$ extends then uniquely to a meromorphic function on all of U in each of the following 3 situations [of which a) is subsumed in b) here, but not always in our generalizations]:

a) F is polar, and the cluster set $C\ell(\varphi,z)$ is distinct from $\mathbb{C}_\infty$ for every $z \in F$.

b) F is polar, and every $z \in F$ has a neighbourhood W in U such that $\mathbb{C}_\infty\setminus\varphi(W\setminus F)$ is non-polar.

c) $C\ell(\varphi,F)$ is polar (in $\mathbb{C}_\infty$), and $\varphi' \not\equiv 0$ in $U\setminus F$. (It follows that F itself is likewise polar.)

Here $C\ell(\varphi,F)$ denotes the cluster set of φ at F, that is the union of all $C\ell(\varphi,z)$ as z ranges over the boundary of

F relative to U.

Our strengthening of the results a) and c) consists essentially in replacing the cluster sets by the corresponding (a priori smaller) *fine* cluster sets. This means that the standard topology on $\mathbb{C}$ for the independent variable is replaced by the Cartan fine topology (the weakest topology making all subharmonic functions continuous). — For a survey of the fine topology and some of its applications see [14].

We shall likewise give a fine topology analogue (not an extension) of the Riesz-Frostman-Nevanlinna-Tsuji theorem concerning boundary cluster sets of a meromorphic function on the disc.

In the case where F reduces to a single point, a) is the classical Casorati-Weierstrass theorem. It was strengthened by Doob [8] in 1965 who replaced the cluster set by the fine cluster set (in the case where F is a singleton).

The result in the situation b) is due to Nevanlinna [21, Kap. V, §4, Satz 3] (1936) for $U = \mathbb{C}_\infty$, and to Kametani [17] (1941) for any U (both for compact F only).

The result in the situation c) is Radó's theorem, essentially as established in [27] (1924) in the case where $C\ell(\varphi,F) = \{0\}$. In that case the result asserts that a continuous function $\varphi : U \to \mathbb{C}$ is holomorphic in U if it is holomorphic in $U\setminus\varphi^{-1}(0)$. This was extended by Cartan [3] (1952) to holomorphic functions of several variables. And this n-dimensional version was further extended by Lelong [20] (1957) who replaced $\varphi^{-1}(0)$ by $\varphi^{-1}(E)$ for any closed polar set $E \subset \mathbb{C}$. The still more general cluster set version (now again in one complex variable, essentially as stated in c) above) is due to Stout [29] (1968). Further extensions were given by Goldstein and Chow [15], Järvi [16], Boboc [2], Cole and Glicksberg [5], Cegrell [4], Oja [22], Riihentaus [28] and Øksendal [23].

The Radó type results announced in the present paper (notably Theorems 2,3,5,7 and 8) can be regarded as generalizations of almost all the results of this type quoted above. The starting point is Cartan's simple proof [3] of the original Radó theorem, based on the subharmonicity of $\log|\varphi|$ in all of U when φ is non-constant and continuous in U and holomorphic in $U\setminus\varphi^{-1}(0)$. This allowed Cartan to reduce Radó's theorem to the removability of closed polar sets as singularity sets for bounded holomorphic functions, that is, the result a) stated above. As noted by Aupetit [1], the same method carries over to give a simple proof of Stout's cluster set version of Radó's theorem stated in c) above.

Briefly speaking the idea in Cartan's proof of Radó's theorem rests on the following potential theoretic property of a holomorphic function φ defined in a domain in $\mathbb{C}^n$: φ is continuous, and $u\circ\varphi$ is harmonic in $\varphi^{-1}(V)$ for any harmonic function u in a domain V in $\mathbb{C}$. Continuous mappings with this property are now called *harmonic morphisms*.

The quoted paper of Boboc [2] (1978) seems to be the first in which the Radó-Stout theorem is extended to harmonic morphisms. In the same spirit we shall in the present paper extend the above results a),b),c) to *finely* harmonic morphisms (§2), that is, the generalization of harmonic morphisms to certain mappings from a finely open subset of one harmonic space X satisfying the axiom of domination into another harmonic space X'. (In particular X and X' could be Riemannian manifolds.) Finely harmonic morphisms were studied by Laine [18], [19], Fuglede [10], [12] and Øksendal [24].

Subsequently we specialize to usual harmonic morphisms. In §3 we apply the results to finely meromorphic and in particular to usual meromorphic functions, and in §4 to holomorphic functions of several variables.

A detailed exposition (with proofs) will appear elsewhere.

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1. Notations and preliminaries.

In sections 1 and 2, X and X' denote two harmonic spaces in the sense of [7] with a countable base for their topology. Except when otherwise stated it is further supposed that X satisfies the *axiom of domination* (axiom D), and that X' is *weakly* $\mathfrak{P}$*-harmonic* (cf. Definition 1 below). These basic hypotheses will not be repeated. They are satisfied e.g. if X and X' are Riemannian manifolds (each endowed with the sheaf of solutions of the Laplace-Beltrami equation). Particular situations are considered in §3 and §4.

Recall that a harmonic space with countable base is a $\mathfrak{P}$-harmonic space if and only if it admits a potential > 0. An open subset of a harmonic space is called a $\mathfrak{P}$-set if it is $\mathfrak{P}$-harmonic as a harmonic subspace. Every union of pairwise disjoint $\mathfrak{P}$-sets of a harmonic space is a $\mathfrak{P}$-set. Every harmonic space admits a covering by $\mathfrak{P}$-sets. A Riemannian manifold is a $\mathfrak{P}$-harmonic space if and only if it has a Green function.

A subset E of a harmonic space is called *polar* if locally (hence actually on every $\mathfrak{P}$-set) there exists a superharmonic function ≥ 0 equal to $+\infty$ in E.

Definition 1. A harmonic space X' will be called *weakly $\mathfrak{P}$-harmonic* if any open subset W' of a component Y' of X' is a $\mathfrak{P}$-set provided that $Y' \setminus W'$ is non-polar (thus in particular when Y' is non-compact and W' is relatively compact in Y').

Every elliptic $\mathfrak{S}$-harmonic space is weakly $\mathfrak{P}$-harmonic, cf. [7, Exerc. 6.2.5]. In particular, every Riemannian manifold is a weakly $\mathfrak{P}$-harmonic space, being a Brelot space (hence elliptic) admitting a superharmonic function > 0 (e.g. the constant 1 which is harmonic).

On the *first* harmonic space X we shall primarily use the *fine topology*, but sometimes also the usual (= initial) topology. Qualifications pertaining to the fine topology are indicated by "fine(ly)". The *fine boundary* of a set $F \subset X$ is denoted by $\partial_f F$. Recall that every polar set F is finely closed and finely isolated and has no finely interior points, whence $\partial_f F = F$.

On the *second* harmonic space X' we shall always employ the *usual topology*. The closure $\overline{A'}$ of a subset A' of X' is always taken in the one-point compactification X'_∞ (= $X' \cup \{\infty\}$ or X') of the locally compact (possibly compact) space X'. (Any other compactification of X' could be used instead with exactly the same results.)

Since the fine topology is generally not even 1st countable, filters must be used in the definition of fine cluster sets.

Definition 2. For any mapping φ of a finely open set $V \subset X$ into X' and for any $x \in \partial_f V$ the fine cluster set $C\ell_f(\varphi,x)$ of φ at x is defined as the set of all points $x' \in X'_\infty$ for which there exists a filter $\mathcal{F}$ on V converging finely to x such that $\varphi(\mathcal{F}) \to x'$ in X'. Equivalently,

$$C\ell_f(\varphi,x) = \bigcap \overline{\varphi(W \cap V)}$$

as W ranges over a fundamental system of fine neighbourhoods of x in X.

Recall that, for a usual open set $U \subset X$, the fine components of U are the same as the usual components of U, see [9, §9.8].

Definition 3. For any mapping φ of a finely open set $V \subset X$ into X' and for any set $F \subset X$ not meeting V the fine cluster set of φ at F (more precisely at $F \cap \partial_f V$) is defined as

$$C\ell_f(\varphi,F) = \bigcup_{x \in F \cap \partial_f V} C\ell_f(\varphi,x).$$

For a mapping φ of a *usual* open set $V \subset X$ into X' and for any point $x \in \partial V$, resp. any set $F \subset X \setminus V$, the usual cluster sets $C\ell(\varphi,x)$ and $C\ell(\varphi,F)$ are defined in analogy with the above definitions, replacing throughout the fine topology on X by the usual topology.

2. Finely harmonic morphisms.

Definition 4. A finely continuous mapping φ of a finely open set $U \subset X$ into X' is called a finely harmonic morphism if $s' \circ \varphi$ is finely hyperharmonic in $\varphi^{-1}(V')$ for every (usual) hyperharmonic function s' in an open set $V' \subset X'$.

If X' has a base (for the usual topology) formed by regular sets then this definition is equivalent to the original one in [10] in which s' was required to be *harmonic* in V' and $s' \circ \varphi$ *finely harmonic* in $\varphi^{-1}(V')$, cf. [12, §2.3]. (In [10], X' was supposed to satisfy axiom D and hence to have a base of regular sets.)

The following theorem extends a) and b) in the introduction.

Theorem 1. *Let* U *denote a finely open subset of* X, *and* F *a polar set contained in* U. *A finely harmonic morphism* $\varphi : U \setminus F \to X'$ *extends to a unique finely harmonic morphism of* U *into* X' *if (and only if) the following two conditions are fulfilled:*

i) $C\ell_f(\varphi, F) \subset X'$.

ii) *For every* $x \in F$ *such that the component* Y' *of* X' *containing* $C\ell_f(\varphi, x)$ *is compact and not a* $\mathfrak{P}$*-set, there is a fine neighbourhood* W *of* x *in* U *such that* $Y' \setminus \varphi(W \setminus F)$ *is non-polar.*

We next bring a corresponding Radó-Stout type result.

Theorem 2. *Let* U *denote a fine domain in* X, *and* F *a relatively finely closed proper subset of* U. *Let* φ *be a finely harmonic morphism, not finely locally constant, of*

$U \setminus F$ *into* X'. *If* $C\ell_f(\varphi, F)$ *is a polar subset of* X', *then* F *is polar (in* X*), and* φ *extends to a unique finely harmonic morphism of* U *into* X'.

The beginning of the proof of the above theorem carries over to establish the following fine topology version of the Riesz-Frostman-Nevanlinna-Tsuji theorem.

Proposition 1. *Suppose that* X' *is* $\mathfrak{P}$*-harmonic. Let* V *denote a fine domain in* X, *and* E *a subset of the fine boundary* $\partial_f V$. *Let* φ *be a non-constant finely harmonic morphism of* V *into* X'. *If* $C\ell_f(\varphi, E)$ *is a polar subset of* X', *then* E *has zero harmonic measure with respect to* V *in the sense that there exists a finely superharmonic function* $s \geq 0$ *on* V *such that*

$$\operatorname*{fine\,lim}_{y \to x} s(y) = +\infty \quad \textit{for every} \quad x \in E.$$

If X is $\mathfrak{P}$-harmonic, the existence of s as stated is equivalent to E being a null set with respect to the (fine) harmonic measure ε_x^{CV} at some (hence any) $x \in V$.

In the next result (to be applied in subsequent sections) we need not assume that X' be weakly $\mathfrak{P}$-harmonic. On the other hand we impose on X' the axiom of polarity, cf. [7, §9.1]. It is a consequence of axiom D and hence fulfilled, in particular, by any Riemannian manifold.

Proposition 2. *Suppose that* X' *satisfies the axiom of polarity and that the points of* X' *are polar. Let* U *and* V

denote finely open subsets of X *such that* $V \subset U$*, and let* φ *be a finely harmonic morphism of* U *into* X'*. If* $U\setminus V$ *is polar (in* X*) then* $\varphi(U)\setminus\varphi(V)$ *is polar (in* X'*).*

We close this section with some indications about the particular case of *usual* harmonic morphisms, cf. [6, §3]. Since we maintain the basic hypotheses on X and X', in particular that X satisfies the axiom of domination, every usual harmonic morphism $\varphi : X \to X'$ is in particular a finely harmonic morphism. In the opposite direction Laine [18] has shown that every finely harmonic morphism $\varphi : X \to X'$ is continuous (in the usual topologies on X and X') and hence is a usual harmonic morphism, provided that X' is $\mathfrak{P}$-harmonic and the points of X' are polar. With this in mind we obtain for example from Theorem 2 the following Radó-Stout type result with *fine* cluster set:

Theorem 3. *Suppose that* X *is connected, that* X' *is* $\mathfrak{P}$*-harmonic, and that the points of* X' *are polar. Let* F *denote a closed proper subset of* X*, and* φ *a harmonic morphism, not locally constant, of* $X\setminus F$ *into* X'*. If* $C\ell_f(\varphi,F)$ *is a polar subset of* X' *then* F *is polar (in* X*), and* φ *extends to a unique harmonic morphism of* X *into* X'*.*

Remarks. 1) The hypothesis that X' be $\mathfrak{P}$-harmonic can be dropped at the expense of replacing $C\ell_f(\varphi,F)$ by the usual cluster set $C\ell(\varphi,F)$, but otherwise not, cf. the example on p.13 below. – This weaker version of Theorem 3 in which the usual cluster set $C\ell(\varphi,F)$ is assumed polar is due to Boboc [2] for slightly more general harmonic spaces X and X' than here (although X' is $\mathfrak{P}$-harmonic in [2]). See also Oja [22] for further generalizations.

2) It would be only apparently more general to allow $C\ell(\varphi,F)$ in Theorem 3 to be just *inner* polar (rather than polar), for $C\ell(\varphi,F)$ is always a K_σ in X'.

3) Suppose that X' is a $\mathfrak{P}$-Brelot space and that all points $x' \in X'$ are *strongly polar* in the sense of [11], that is, every non-zero potential on X' harmonic off $\{x'\}$ should take the value $+\infty$ at x'. (Every Riemannian manifold X' is a Brelot space, and the points of X' are strongly polar if $\dim X' > 1$.) Under this hypothesis, every non-constant harmonic morphism of X (connected) into X' is an *open* mapping (with respect to the usual topologies on X and X'), see [11]. In Theorem 3 above the hypothesis that $C\ell_f(\varphi,F)$ be polar can then be weakened in the spirit of Øksendal [23], [24] as follows: Let V denote a component of $X\setminus F$ on which φ is not constant, and suppose that $C\ell_f(\varphi,F)$ (or just that $C\ell_f(\varphi|V, U\setminus V)$) is polar with respect to the domain $\varphi(V)$ according to the following definition:

Definition 5. Let U' be a $\mathfrak{P}$-set in X'. A set $F' \subset X'$ is said to be polar with respect to U' if $F' \cap U'$ is polar, while $F' \cap \partial U'$ has zero harmonic measure with respect to U'.

3. Applications to finely holomorphic and finely meromorphic functions.

First an easy extension to Riemann surfaces of the notion of finely holomorphic function defined on a finely open subset of $\mathbb{C}$ and taking values in $\mathbb{C}$, cf. [13] and the survey [14].

We only consider connected Riemann surfaces.

A Riemann surface X becomes a Brelot harmonic space with a countable base when the sheaf of harmonic functions is taken as the functions which locally are real parts of holomorphic functions from (open subsets of) X into $\mathbb{C}$. Equivalently, the harmonic functions on an open subset of X are the solutions on that set to the Laplace-Beltrami equation with respect to a Riemannian metric on X chosen — as it may be done — so that every holomorphic function (on any open subset of X) becomes complex harmonic.

Since the constant functions are harmonic, a Riemann surface X is thus an $\mathfrak{S}$-Brelot space. It is known that axiom D holds for any Riemann surface, and that its points are polar.

Now consider two Riemann surfaces X, X'. A mapping φ of a finely open subset U of X into X' is termed *finely holomorphic* if φ is finely continuous (i.e., continuous from U with the fine topology to X' with the usual topology) and if moreover $z' \circ \varphi \circ z^{-1}$ is finely holomorphic (on the finely open subset $z(V \cap \varphi^{-1}(V'))$ of $\mathbb{C}$ into $\mathbb{C}$) for every choice of complex coordinates z, z' on coordinate neighbourhoods V, V' in X, X', respectively.

Every finely holomorphic mapping $\varphi : U \to X'$ as above is, in particular, a finely harmonic morphism. If U is a fine domain and φ is non-constant then the pre-image $\varphi^{-1}(a')$ of any point $a' \in X'$ is not only polar, but even *countable*, cf. [13, §15] for the typical case $X = X' = \mathbb{C}$.

Via local coordinates z, z' for X, X' the following proposition reduces immediately to the corresponding result for the case $X = X' = \mathbb{C}$ which in turn is contained in [13, Cor. 3].

Proposition 3. *Let* φ *denote a finely continuous mapping of a finely open subset* U *of a Riemann surface* X *into a Riemann surface* X'. *If* φ *is finely holomorphic in* $U\setminus F$ *for some polar set* F *in* X *then* φ *is finely holomorphic in all of* U.

It follows from this proposition that, in the case of Riemann surfaces X, X', one may replace the term "finely harmonic morphism" by "finely holomorphic mapping" in the results of §2.

While retaining X as an arbitrary Riemann surface we shall henceforth specialize to $X' = \mathbb{C}_\infty$, the Riemann sphere. Finely holomorphic mappings of a finely open set $U \subset X$ into $\mathbb{C}$ or $\mathbb{C}_\infty$ will be called *finely holomorphic functions* or *finely meromorphic functions*, respectively.

Consider a finely meromorphic function $\varphi : U \to \mathbb{C}_\infty$. For any point $a \in U$ and any coordinate $z : V \to \mathbb{C}$ on X near a such that $z(a) = 0$ there is a unique integer n such that φz^{-n} extends by fine continuity to a finely holomorphic function $U \cap V \to \mathbb{C}$ taking a non-zero value at the given point a. If $n < 0$ then a is called a *pole* of order $|n|$ for φ.

On a *usual* open subset U of a Riemann surface X the holomorphic and the finely holomorphic functions $U \to \mathbb{C}$ are the same, cf. [13, p.63] for the typical case $X = \mathbb{C}$.

Proposition 4. *A finely meromorphic function* $\varphi : X \to \mathbb{C}_\infty$ *is meromorphic if* $\varphi(X) \neq \mathbb{C}_\infty$, *or more generally if every point of* X *has a (usual) neighbourhood* W *such that* $\varphi(W) \neq \mathbb{C}_\infty$.

Example. In $X = \mathbb{C}$ write $z_n = 2^{-n}$, and choose a sequence of constants a_n tending to 0 sufficiently rapidly as $n \to \infty$ so that $\Sigma |a_n| < \infty$ and the series $\Sigma\, a_n/(z-z_n)$ converges uniformly in some fine neighbourhood of 0. The series converges locally uniformly off 0 and the points z_n, hence determines altogether a finely meromorphic function φ in all of $\mathbb{C}$, cf. [13, p.74]. (Earlier this example was used by Doob [8, p.125 f.].) Clearly φ is *not* meromorphic in the whole of $\mathbb{C}$.

From Theorem 1 combined with Propositions 2 and 3 we immediately obtain the following result of Nevanlinna-Kametani type for finely meromorphic functions:

Theorem 4. *Let* U *denote a finely open subset of a Riemann surface* X, *and* F *a polar subset of* U. *A finely meromorphic function* φ *on* $U \setminus F$ *extends to a unique finely meromorphic function* φ^* *on* U *if (and only if) every point of* F *has a fine neighbourhood* W *in* U *such that* $\mathbb{C}_\infty \setminus \varphi(W \setminus F)$ *is non-polar. If even* $\mathbb{C}_\infty \setminus \varphi(U \setminus F)$ *is non-polar then* φ^* *is globally finely meromorphic, that is, a quotient of two finely holomorphic functions.*

Remark. This result (for the case $X = \mathbb{C}$) was announced in

[14]. The typical case where F reduces to a single point $a \in U$ was obtained in conversation with T.J. Lyons and A.G. O'Farrell. — In the case of a usual holomorphic function $\varphi : X \setminus \{a\} \to \mathbb{C}$ such that $\mathbb{C} \setminus \varphi(W \setminus \{a\})$ is non-polar for some *fine* neighbourhood W of a we conclude that φ extends to a usual meromorphic function in all of X (with a as its only possible pole), viz. to a finely meromorphic function in X with at most one pole. — This consequence of Theorem 4 was pointed out by Lyons (in a letter to the author). It is stronger than the otherwise similar result of Casorati-Weierstrass type obtained by Doob [8, Theorem 7.3] (prior to the appearence of finely harmonic or finely holomorphic functions).

<u>Corollary</u>. *Let* F *denote a polar relatively closed subset of an open set* U *in* X. *A holomorphic function* $\varphi : U \setminus F \to \mathbb{C}$ *extends to a unique holomorphic function on* U *if (and only if)* $C\ell_f(\varphi, F) \subset \mathbb{C}$.

Another consequence of Theorem 4 involves the irregular part of the *boundary* of an irregular open set.

<u>Corollary</u>. *Let* φ *be meromorphic in a usual open set* $V \subset X$ *and let* F *denote the set of <u>irregular</u> points for the Dirichlet problem in* V. *If* $\mathbb{C}_\infty \setminus \varphi(V)$ *is non-polar then* φ *extends by fine continuity to a unique <u>finely</u> meromorphic function* φ^* *on the <u>finely</u> open set* $V \cup F \subset X$.

Like in Theorem 4 it suffices to suppose that every point of F has a fine neighbourhood W in X such that $\varphi(W \cap V)$

is not co-polar. — Note that, in the affirmative case, very precise information is available as to the behaviour of φ^* in a suitable fine neighbourhood of each point of F, cf. [13, théorème 11] (for a typical case). — To derive this corollary from Theorem 4, observe that φ is, in particular, finely meromorphic in $V = U \setminus F$, and that $U := V \cup F$ is finely open, the irregular points for V being precisely the finely isolated points of $\complement V$.

Next we derive from Theorem 2 and Proposition 3 a Radó-Stout type theorem for finely meromorphic functions:

Theorem 5. *Let* U *denote a fine domain in a Riemann surface* X*, and* F *a relatively finely closed proper subset of* U*. Let* φ *be finely meromorphic in* $U \setminus F$ *with* $\varphi' \not\equiv 0$*, and suppose that* $C\ell_f(\varphi, F)$ *is polar (in* $\mathbb{C}_\infty$*). Then* F *is polar (in* X*), and* φ *extends to a unique finely meromorphic function on* U.

For *bounded* finely holomorphic functions a stronger resul has quite recently been obtained by Øksendal [24].

For *usual* holomorphic functions we obtain the following

Corollary. *Let* F *denote a relatively closed proper subset of a domain* U *in* X*. If* φ *is holomorphic in* $U \setminus F$ *with* $\varphi' \not\equiv 0$ *and if* $C\ell_f(\varphi, F)$ *is polar and contained in* $\mathbb{C}$*, then* F *is polar and* φ *extends to a unique holomorphic function in* U.

In fact, the finely meromorphic extension of φ to U omits the value ∞ and is therefore holomorphic, by

Proposition 4.

Remark. The example on p.13 shows that one cannot in general in the meromorphic case omit the word "fine(ly)" (4 times) in Theorem 5 and still keep the *fine* cluster set $C\ell_f(\varphi,F)$. But if one also replaces the fine cluster set by the usual cluster set then one recovers the Lelong-Stout extension of Radó's theorem mentioned in c) in the introduction.

4. Applications to holomorphic functions of several variables.

Let us specialize the results of §2 to holomorphic mappings $X \to X'$ where X is a domain in $\mathbb{C}^n$, $n \geq 1$, or more generally a Kähler manifold of complex dimension n, and where X' is $\mathbb{C}$ or $\mathbb{C}_\infty$, or just any Riemann surface. Then X and X' are connected $\mathfrak{S}$-Brelot space (hence weakly $\mathfrak{P}$-harmonic) with countable base satisfying axiom D (in particular the axiom of polarity), and all points of X or X' are strongly polar.

Every holomorphic mapping $\varphi : X \to X'$ is clearly a harmonic morphism, in particular a finely harmonic morphism. It is well known that a continuous mapping $\varphi : X \to X'$ which is holomorphic off some closed polar subset of X is holomorphic in all of X, cf. Lelong [20]. With these circumstances in mind we obtain from the results of §2 the following Theorems 6 and 7 generalizing the classical results a), b) and c) from the introduction to several complex variables.

Theorem 6. *Let* F *denote a closed polar subset of* X. *A* *holomorphic mapping* $\varphi : X\setminus F \to X'$ *extends to a unique holomorphic mapping of* X *into* X' *in each of the following* 3 *situations:*

a) X' *is* $\mathfrak{P}$*-harmonic and* $C\ell_f(\varphi,F) \subset X'$.

b) X' *is non-compact and* $C\ell(\varphi,F) \subset X'$.

c) X' *is compact and every point of* X *has a neighbourhood* W *in* X *such that* $X'\setminus\varphi(W\setminus F)$ *is non-polar.*

Remark. If $n = 1$ (so that X too is a Riemann surface), $C\ell(\varphi,F)$ may be replaced by $C\ell_f(\varphi,F)$ in b). This is presumably no longer true when $n > 1$. – Even if $n = 1$, W cannot be allowed to be just a *fine* neighbourhood in c), as shown by the example on p.13.

Theorem 7. *Let* F *denote a closed proper subset of* X, *and* $\varphi : X\setminus F \to X'$ *a holomorphic mapping, not locally constant. In either of the* 3 *situations* a), b) *or* c) *in Theorem* 6, *suppose that* $C\ell_f(\varphi,F)$ *is polar (in* X'*). Then* F *is pluripolar, and* φ *extends to a unique holomorphic mapping of* X *into* X'.

Remarks. 1) Let V denote a component of $X\setminus F$ on which φ is not constant. Instead of supposing that $C\ell_f(\varphi,F)$ be polar it suffices in Theorem 7 in the situation a) where X' is $\mathfrak{P}$-harmonic to assume that $C\ell_f(\varphi,F)$ (or even just $C\ell_f(\varphi|V, U\setminus V)$) be contained in X' and polar with respect to the domain $\varphi(V)$, cf. Def. 5 and Remark 3 to Theorem 3.

2) The fact that F must *pluripolar* (rather than just

polar) was noted by Riihentaus [28].

3) Let $X' = \mathbb{C}$, so that we are in the situation b). The hypothesis that $C\ell(\varphi,F)$ be contained in $\mathbb{C}$ (that is, ∞ is not a cluster value) may be replaced in Theorem 6 b) and hence in Theorem 7 b) by the priori weaker hypothesis that φ be of *Hardy class* H^p (see Definition 6 below) in $W\setminus F$ for some open neighbourhood W of each point of ∂F, and for some p, $0 < p < +\infty$. In the case $n = 1$ this goes back to Parreau [26] as to Theorem 6 b); and to Goldstein and Chow [15] (cf. also Osada [25]) as to Theorem 7 b), though with the *usual* cluster set $C\ell(\varphi,F)$ being assumed polar. The alternative proof of Parreau's theorem given by Yamashita [30] carries over to the n-dimensional case.

Definition 6. A holomorphic function $\varphi : U \to \mathbb{C}$ (U open in X) is said to be of class H^∞ if φ is bounded, and of class H^p $(0 < p < +\infty)$ if the subharmonic function $|\varphi|^p$ has a superharmonic (hence also a harmonic) majorant in U.

For U = the unit disc in $\mathbb{C}$ this definition agrees with the classical one. For the sake of simplicity we enunciate below the Hardy class version of Theorems 6 b) and 7 b) only in the case where φ is *globally* of class H^p:

Proposition 5. *Let* F *denote a closed polar subset of* X, *and let* $\varphi : X\setminus F \to \mathbb{C}$ *be holomorphic of class* H^p *for some* p, $0 < p \le +\infty$. *Then* φ *extends to a unique holomorphic function on* X, *likewise of class* H^p.

Theorem 8. *Let* F *denote a closed proper subset of* X, *and* *let* $\varphi : X \setminus F \to \mathbb{C}$ *be holomorphic of class* H^p *for some* p, $0 < p \leq +\infty$ *If* $\varphi' \not\equiv 0$ *and if* $C\ell_f(\varphi, F)$ *is polar (in* $\mathbb{C}_\infty$*) then* F *is pluripolar, and* φ *extends to a unique holomorphic function on* X*, likewise of class* H^p.

Remark. In the case $p = +\infty$ Proposition 5 is due to Lelong [20], and Theorem 8 to Cegrell [4] except for our use of the fine cluster set.

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